Constructions of new matroids and designs over $\mathbb{F}_q$

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Matroid: a pair \((E, r)\) with

- \(E\) finite set;
- \(r : 2^E \to \mathbb{N}_0\) a function, the rank function, with for all \(A, B \in E\):
  
  (r1) \(0 \leq r(A) \leq |A|\)
  (r2) If \(A \subseteq B\) then \(r(A) \leq r(B)\).
  (r3) \(r(A \cup B) + r(A \cap B) \leq r(A) + r(B)\) (semimodular)
Example

$$
\begin{pmatrix}
0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0
\end{pmatrix}
$$

Example

But: most matroids don’t come from a matrix or graph.
**Independent set**: subset with rank equal to cardinality

**Circuit**: minimal dependent subset

**Flat**: subset such that adding an element increases the rank

Matroids are completely determined by their independent sets, circuits, and flats.
Example (Fano plane)

\[ E = \{1, 2, 3, 4, 5, 6, 7\} \]

Independent sets:
\[ \emptyset, \text{ points, pairs of points, } 3 \text{ points not on a line} \]

Flats:
\[ \emptyset, \text{ points, lines, } E \]

Circuits:
\[ \text{lines, 4 points with no 3 on a line} \]
A perfect matroid design (PMD) is a matroid where all flats of the same rank have the same size.

Example

- Finite projective space and its truncations
- Finite affine space and its truncations
- Steiner systems
- Triffids, coming from finite commutative Moufang loops
Theorem (Murty, Young, Edmonds, 1970)

Given a PMD, we can make the following designs:

- All flats of cardinality \( j \) form the set of blocks of a design.
- All independent sets of cardinality \( j \) form the set of blocks of a design.
- All circuits of cardinality \( j \) form the set of blocks of a design.
Example (Fano plane)

\[ E = \{1, 2, 3, 4, 5, 6, 7\} \]

Independent sets:
\(\emptyset\), points, pairs of points,
3 points not on a line

Flats:
\(\emptyset\), points, lines, \(E\)

Circuits:
lines, 4 points with no 3 on a line
$q$-Analogues

<table>
<thead>
<tr>
<th>lattice</th>
<th>Boolean subspace lattice of $\mathbb{F}_q^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>atom</td>
<td>element</td>
</tr>
<tr>
<td>height</td>
<td>1-dim subspace</td>
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<tr>
<td># atoms</td>
<td>size</td>
</tr>
<tr>
<td>meet $\land$</td>
<td>dimension</td>
</tr>
<tr>
<td>$n$</td>
<td>$\frac{q^n-1}{q-1}$</td>
</tr>
<tr>
<td>join $\lor$</td>
<td>intersection</td>
</tr>
<tr>
<td>union</td>
<td>sum</td>
</tr>
</tbody>
</table>

From $q$-analogue to ‘normal’: let $q \to 1$. 
$q$-Matroid: a pair $(E, r)$ with

- $E$ finite dimensional vector space;
- $r : \{\text{subspaces of } E\} \rightarrow \mathbb{N}_0$ a function, the rank function, with for all $A, B \subseteq E$:
  
  (r1) $0 \leq r(A) \leq \dim A$
  
  (r2) If $A \subseteq B$ then $r(A) \leq r(B)$.
  
  (r3) $r(A + B) + r(A \cap B) \leq r(A) + r(B)$ (semimodular)
Theorem (J. & Pellikaan, 2018)
Every \( \mathbb{F}_{q^m} \)-linear rank metric code gives a \( q \)-matroid.

Proof.
Let \( E = \mathbb{F}_q^n \) and \( G \) be a generator matrix of the code.
Let \( A \subseteq E \) and \( Y \) a matrix whose columns span \( A \).

Then \( r(A) = \text{rk}(GY) \) satisfies the axioms (r1),(r2),(r3). \( \square \)
A $q$-PMD is a $q$-matroid where all flats of the same rank have the same dimension.

Example
All $q$-Steiner systems are $q$-PMD’s, where the blocks of the $q$-Steiner system are the maximal proper flats of the $q$-PMD.
Theorem (Byrne, Ceria, Ionica, J., Saçıkara, 2020)

Given a $q$-Steiner system and viewing it as a $q$-PMD, we can make the following subspace designs:

- All flats of dimension $j$ form the set of blocks of a design.
- All independent spaces of dimension $j$ form the set of blocks of a design.
- All circuits of dimension $j$ form the set of blocks of a design.

Corollary

There exists a 2-$(13, 4, 5115; 2)$ design.
Theorem (Byrne, Ceria, Ionica, J., Saçıkara, 2020)

The subspace designs obtained from a $q$-Steiner system in the previous theorem have the same automorphism group as the $q$-Steiner system.

What we need: more $q$-Steiner systems!
Thank you for your attention!